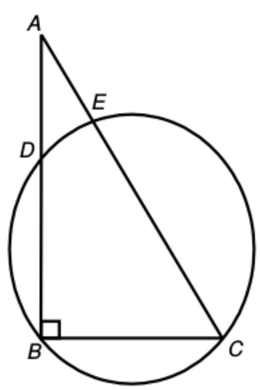
(1) Let the sequence $\{a_n\}_{n\geq 1}$ be defined by

$$a_n = \tan(n\theta),$$

where $\tan(\theta) = 2$. Show that for all $n$, $a_n$ is a rational number which can be written with an odd denominator.

(2) Consider a circle of radius 6 as given in the diagram below. Let $B$, $C$, $D$ and $E$ be points on the circle such that $BD$ and $CE$, when extended, intersect at $A$. If $AD$ and $AE$ have length 5 and 4 respectively, and $DBC$ is a right angle, then show that the length of $BC$ is $\frac{12+9\sqrt{15}}{5}$.




(3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function given by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ e^{(x^{10}-1)} + (x-1)^2 \sin\left(\frac{1}{x-1}\right) & \text{if } x \neq 1. \end{cases}$$

(a) Find $f'(1)$.

(b) Evaluate $\displaystyle\lim_{u\to\infty} \left[100\,u - u\sum_{k=1}^{100} f\left(1+\frac{k}{u}\right)\right]$.

(4) Let $S$ be the square formed by the four vertices $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$. Let the region $R$ be the set of points inside $S$ which are closer to the centre than to any of the four sides. Find the area of the region $R$.

P.T.O.

(5) Let $g : \mathbb{N} \to \mathbb{N}$ with $g(n)$ being the product of the digits of $n$.
   (a) Prove that $g(n) \leq n$ for all $n \in \mathbb{N}$.
   (b) Find all $n \in \mathbb{N}$, for which $n^2 - 12n + 36 = g(n)$.

(6) Let $p_1, p_2, p_3$ be primes with $p_2 \neq p_3$, such that $4 + p_1p_2$ and $4 + p_1p_3$ are perfect squares. Find all possible values of $p_1, p_2, p_3$.

(7) Let $A = \{1, 2, \ldots, n\}$. For a permutation $P = (P(1), P(2), \cdots, P(n))$ of the elements of $A$, let $P(1)$ denote the first element of $P$. Find the number of all such permutations $P$ so that for all $i, j \in A$:
- if $i < j < P(1)$, then $j$ appears before $i$ in $P$; and
- if $P(1) < i < j$, then $i$ appears before $j$ in $P$.

(8) Let $k, n$ and $r$ be positive integers.
   (a) Let $Q(x) = x^k + a_1x^{k+1} + \cdots + a_nx^{k+n}$ be a polynomial with real coefficients. Show that the function $\frac{Q(x)}{x^k}$ is strictly positive for all real $x$ satisfying
$$0 < |x| < \frac{1}{1 + \sum_{i=1}^{n} |a_i|}.$$
   (b) Let $P(x) = b_0 + b_1x + \cdots + b_rx^r$ be a non-zero polynomial with real coefficients. Let $m$ be the smallest number such that $b_m \neq 0$. Prove that the graph of $y = P(x)$ cuts the $x$-axis at the origin (i.e. $P$ changes sign at $x = 0$) if and only if $m$ is an odd integer.